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Non-conformal harmonic maps into the 3-sphere

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We present two transforms of non-conformal harmonic maps from a surface into the 3-sphere. With these transforms one can construct from one non-conformal harmonic map a sequence of non-conformal harmonic maps. We also discuss the correspondence between non-conformal harmonic maps into the 3-sphere, H-surfaces in Euclidean 3-space and almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$.

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1. Introduction

In this paper we give a summary of the main results in [5]. A map f from a Riemann surface S into the unit 3-sphere S^3 is harmonic if it satisfies $\Delta f + |df|^2 f = 0$. The map is conformal if it preserves the conformal structure on S, i.e. $\langle \partial f, \partial f \rangle = 0$ where ∂ denotes $\frac{\partial}{\partial z}$. If the map is both harmonic and conformal it is a minimal immersion of the surface in the 3-sphere. In this paper we consider non-conformal maps, i.e. $\langle \partial f, \partial f \rangle \neq 0$. We will present two transformations which turn a non-conformal harmonic map from S to S^3 into a new non-conformal harmonic map from S to S^3. Using these two transforms one can define a sequence $\{f^p \,|\, p \in \mathbb{Z}\}$ of non-conformal harmonic maps from S into S^3 where $f^0 = f$.

The transforms are natural generalizations of the polar construction for superconformal minimal surfaces in odd-dimensional spheres (see [3, 6]) and have a nice geometrical interpretation. The transforms of harmonic maps were inspired by the work [4]. In this paper Bolton and Vrancken described transforms of minimal surfaces in S^5 with non-circular ellipse of curvature. Antic and Vrancken [1] generalized these transforms for superconformal minimal surfaces in odd-dimensional spheres S^{2n+1} whose $(n-2)$ higher order ellipses of curvature are circles.

Our original motivation to investigate the transforms of harmonic maps is the study of almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$. In the last section we will explain the relation between almost complex surfaces in $S^3 \times S^3$, H-surfaces in $\mathbb{R}^3$ and harmonic maps into S^3. As a corollary one can associate to one almost complex surface in $S^3 \times S^3$ with a non-vanishing differential a whole sequence of almost complex surfaces.

2. Harmonic Maps to S^3

We will use a quaternionic language to describe harmonic maps into the 3-sphere. The ring of quaternions $\mathbb{H}$ can be identified with the vector space $\mathbb{R}^4$. Quaternions are real linear combinations of the basis elements 1, e_1, e_2 and e_3 and their multiplication is determined by $e_1^2 = e_2^2 = e_3^2 = e_1e_2e_3 = -1$. A quaternion that is a linear combination of e_1, e_2 and e_3 is called an imaginary quaternion. The product of two imaginary quaternions α and β is given by $\alpha\beta = -\langle\alpha,\beta\rangle + \alpha \times \beta$ where $\langle\,,\,\rangle$ is the Euclidean inner product and $\times$ is the usual vector product on $\mathbb{R}^3$.

Quaternions are very useful to describe the 3-sphere and its tangent spaces. The 3-sphere S^3 is the set of unit quaternions $\{p \in \mathbb{H} \mid \|p\| = 1\}$. One can show that $p\alpha$ is orthogonal to p for every imaginary quaternion α, thus the tangent space of S^3 at p is

$$T_pS^3 = \{p\alpha \mid \alpha \in \operatorname{Im}\mathbb{H}\}. \tag{1}$$

On the surface S we will use complex coordinates, so in order to describe the complexified tangent vectors we will use complexified quaternions $\mathbb{H} \otimes \mathbb{C} = \mathbb{H} \oplus i\mathbb{H}$. The element i must be distinguished from $e_1 \in \mathbb{H}$. The complex bilinear extension of the Euclidean metric and vector product are also denoted by $\langle\,,\,\rangle$ and $\times$. The conjugate of a complexified quaternion $p_1 + ip_2$ is equal to $\overline{p_1 + ip_2} = p_1 - ip_2$ and is denoted with a bar.

Now consider a non-conformal harmonic map $f : S \to S^3 \subset \mathbb{H}$ from a surface S into the 3-sphere S^3. Choose a local complex coordinate $z = x + iy$ on S. Write $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ respectively as ∂ and $\bar{\partial}$. Let f_x and f_y be the derivatives of f with respect to x and y respectively. We introduce the $\mathbb{H} \otimes \mathbb{C}$-valued function

$$f_1 = \partial f.$$

Since $\langle f, f\rangle = 1$, it follows that $\langle f, f_1\rangle = 0$. Harmonicity of f means that $\partial\bar{\partial} f = -|\partial f|^2 f$ and non-conformality says that $\langle f_1, f_1\rangle$ is non-zero. By the harmonicity and non-conformality the function $\langle f_1, f_1\rangle$ is holomorphic and non-zero. Therefore there exists a complex coordinate z such that $\langle f_1, f_1\rangle = -1$. We will call such a coordinate an *adapted complex coordinate for* f.

By (1) there exist functions α and β with values in $\operatorname{Im}\mathbb{H}$ such that $f_x = f\alpha$ and $f_y = f\beta$. It follows from $\langle f_1, f_1\rangle = -1$ that $\langle\alpha,\alpha\rangle - \langle\beta,\beta\rangle = -4$ and $\langle\alpha,\beta\rangle = 0$, so there is a non-negative smooth function ϕ such that

$$|\alpha| = 2\sinh\phi, \quad |\beta| = 2\cosh\phi.$$

Note that β vanishes nowhere, but α can be zero. At points where α is not zero the vectors f_1 and $\bar{f}_1$ are linearly independent and ϕ is positive. In the following we assume that we are working on the open subset U of S where $\alpha \neq 0$. At a point of U

define N as the real unit vector in the direction of $f(\alpha \times \beta)$. Then N is orthogonal to $\{f, f_1, \bar{f}_1\}$ and $|f(\alpha \times \beta)|^2 = 4\sinh^2 2\phi$, hence $f(\alpha \times \beta) = \pm 2\sinh 2\phi N$. For definiteness we choose $N = \frac{1}{2}\operatorname{csch} 2\phi f(\alpha \times \beta)$. We now have a complex moving frame $\mathcal{F} = \{f, f_1, \bar{f}_1, N\}$ for f on the set U. The moving frame equations for $\mathcal{F}$ are

$$\begin{aligned} \partial f &= f_1, \\ \partial f_1 &= f + 2\partial\phi(\coth 2\phi f_1 + \operatorname{csch} 2\phi \bar{f}_1) + \mu N, \\ \partial \bar{f}_1 &= -\cosh 2\phi f, \\ \partial N &= -\mu \operatorname{csch} 2\phi(\operatorname{csch} 2\phi f_1 + \coth 2\phi \bar{f}_1), \end{aligned} \tag{2}$$

where $\mu = \langle \partial f_1, N\rangle$. The complex function μ measures the rate at which the image of f is pulling away from the great 2-sphere tangent to the image of f. The compatibility conditions $\partial\bar{\partial}\mathcal{F} = \bar{\partial}\partial\mathcal{F}$ for the frame $\mathcal{F}$ are

$$\begin{aligned} 2\partial\bar{\partial}\phi &= -\sinh 2\phi + |\mu|^2 \operatorname{csch} 2\phi, \\ \bar{\partial}\mu &= -2\bar{\mu}\partial\phi \operatorname{csch} 2\phi. \end{aligned}$$

If f is a map into a great 2-sphere, then μ vanishes and the above compatibility condition for ϕ becomes the sinh-Gordon equation.

3. The Transforms

Let $f : S \to S^3$ be a non-conformal harmonic map. In this section we show how to associate to f two new non-conformal harmonic maps f^+ and f^- from S into S^3.

Fix a point $p \in S$ and consider the vectors

$$\pm\sin\theta \frac{f\beta}{|f\beta|} + \cos\theta N \tag{3}$$

in $T_{f(p)}S^3$, where θ is chosen such that $\cos\theta = |\alpha|/|\beta| = \tanh\phi$ and $\sin\theta = \operatorname{sech}\phi$. The ellipse E with $f\alpha$ and $f\beta$ as minor and major semi-axes is the image of a circle in the tangent plane to S at p under df (see Fig. 1). The cosine $\cos\theta$ is the ratio

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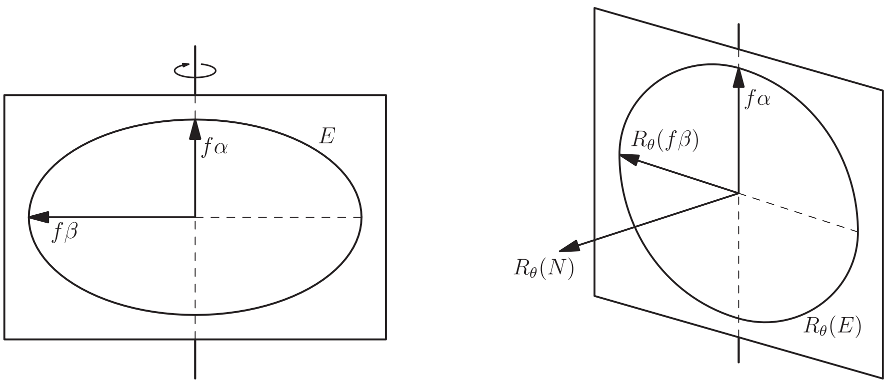

Fig. 1. The ellipse E and the rotated ellipse $R_\theta(E)$.

between the lengths of the minor and major axes of this ellipse. It is a measure for the eccentricity of E as well as for the non-conformality of f. The vectors above have the following geometrical meaning. Let R_θ be the rotation of $T_{f(p)}S^3$ about the minor axis of E through the angle θ. Then the orthogonal projection of the rotated ellipse $R_\theta(E)$ onto the plane containing E is a circle. Of course, the same holds for the rotation $R_{-\theta} = R_\theta^{-1}$. The vectors above are the images of the unit normal N under the rotations R_θ and $R_{-\theta}$ (see Fig. 1).

We can rewrite the vectors in (3) as

$$f^+ = \frac{i}{2}\operatorname{sech}^2\phi(f_1 - \bar{f}_1) + \tanh\phi N,$$

$$f^- = -\frac{i}{2}\operatorname{sech}^2\phi(f_1 - \bar{f}_1) + \tanh\phi N.$$

By varying the point p, we can regard f^+ and f^- as maps from S to S^3 and we call them the $(+)$*transform* and $(-)$*transform* of f. If f were conformal, that is if $|\alpha| = |\beta|$ everywhere, the expressions (3) would still make sense. In fact, we get $\theta = 0$ and recover the polar surface of the minimal surface f, see [6].

The following theorems can be proven using the moving frame equations (2).

Theorem 1. *Let $f : S \to S^3$ be a non-conformal harmonic map from a Riemann surface S into the 3-sphere. Then the transforms f^+ and f^- are also non-conformal harmonic maps from S to S^3. Furthermore, an adapted complex coordinate for f is also an adapted complex coordinate for f^+ and f^-.*

Theorem 2. *The $(+)$transform and $(-)$transform are mutual inverses*:

$$(f^+)^- = (f^-)^+ = f.$$

By Theorems 1 and 2 one can associate to a non-conformal harmonic map $f : S \to S^3$ a sequence $\{f^p \mid p \in \mathbb{Z}\}$ of such harmonic maps by defining $f^0 = f$ and $f^{p+1} = (f^p)^+$ and $f^{p-1} = (f^p)^-$ for every integer p. Moreover, z is an adapted complex coordinate for every map in the sequence.

4. Almost Complex Surfaces in $S^3 \times S^3$

In this section we discuss the relation between harmonic maps $f : S \to S^3$, H-surfaces in $\mathbb{R}^3$ and almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$. For more details on the nearly Kähler structure of $S^3 \times S^3$ and almost complex surfaces in $S^3 \times S^3$ we refer to [2].

A H-surface X in $\mathbb{R}^3$ is a surface satisfying $X_{xx} + X_{yy} = 2HX_x \times X_y$ where $z = x + iy$ is a complex coordinate and H is a non-zero constant. The definition is independent of the choice of the coordinate z. It follows from the defining equation that $\langle X_z, X_z\rangle dz^2$ is a holomorphic differential. If $\langle X_z, X_z\rangle dz^2 = 0$, the surface X is conformal and is a surface in $\mathbb{R}^3$ of constant mean curvature (CMC) H. In [2] it was proven that H-surfaces in $\mathbb{R}^3$ correspond to almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$. An almost complex surface in $S^3 \times S^3$ also admits

Table 1. Correspondence between the surfaces.

General case	Harmonic map $S \to S^3$	H-surface $S \to \mathbb{R}^3$	Almost complex surface $S \to S^3 \times S^3$
Conformal case	Minimal surface	CMC surface	Almost complex surface with $\Lambda dz^2 = 0$

a natural holomorphic differential Λdz^2. The differentials on the H-surface and its associated almost complex surface in $S^3 \times S^3$ are related by $\Lambda = e^{i\frac{\pi}{3}}\langle X_z, X_z\rangle$. Therefore a CMC surface in $\mathbb{R}^3$ corresponds to an almost complex surface in $S^3 \times S^3$ with $\Lambda dz^2 = 0$.

The Lawson correspondence [6] states that minimal surfaces in S^3 correspond to CMC surfaces in $\mathbb{R}^3$. The non-conformal analogue of this theorem [5, Proposition 5.2] says that harmonic maps into S^3 correspond to H-surfaces in $\mathbb{R}^3$. We summarize all the correspondences in Table 1.

Theorems 1 and 2 and the correspondences directly give the next corollary.

Corollary 3. *To a simply connected almost complex surface with non-zero differential we can associate a sequence of almost complex surfaces with non-zero differentials. Similarly a simply connected H-surface in $\mathbb{R}^3$ with a non-zero differential induces a sequence of H-surfaces with non-zero differentials.*

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